Coupled Oscillators

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Purpose:

To observe what happens when two oscillators interact.

Equipment:

Agilent 33120A Function Generator

Theory:

It is quite common for an oscillatory system to be coupled to another one, accidentally or deliberately. Technological examples include the electrical and electromechanical filters used in communications, the strings and sounding boards of musical instruments, mechanical vibration dampers, and loudspeakers in "bass reflex" enclosures. As we will see here, interaction with another system can substantially alter the frequency response of both oscillators.

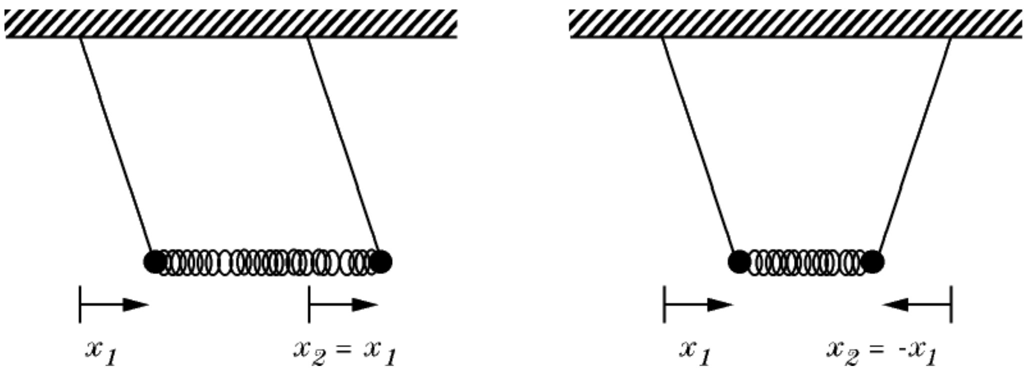

Fig. 1 Normal modes of coupled pendula.

The classic example, illustrated in many texts, is two pendula joined by a spring. If one pendulum is started swinging with small amplitude, the other slowly builds up amplitude as the spring feeds energy from the first into the second. Then the energy flows back into the first and the cycle repeats. The general behavior can be quite complicated, depending on the initial excitation as well as the system parameters, but a particularly simple situation can be set up for two identical pendula. If you start the two swinging together they will continue to swing in unison at their natural frequency. Alternatively, if they are started exactly out of phase (swinging in opposite directions), they will maintain this motion, but at a higher frequency than they would oscillate if uncoupled. These two possibilities are called the normal modes of the system, and are illustrated in Fig. 1. When the pendula are not identical there are still two normal modes, but the motions are more complicated and neither mode is at the uncoupled frequency. We will see this in detail later.

For convenience, we will study the electronic analog of a pendulum. In a previous experiment you observed the behavior of a series RLC circuit, with and without a driving force. In this experiment you will observe the response of a pair of coupled RLC circuits, as idealized in Fig. 2. The components L_aC_a form one LC resonant circuit and the components L_bC_b form the other. The capacitor C_c couples the two circuits. In the limit as $C_c \rightarrow \infty$ the two circuits are uncoupled, since the capacitor could then be replaced by a wire as far as AC signals are concerned. The other limit, $C_c \rightarrow 0$, corresponds to strong coupling, and the circuits behave like a single LC circuit with $L = L_a + L_b$ and $1/C = 1/C_a + 1/C_b$. The intermediate range of relatively weak coupling is more interesting. As we will see below, this situation leads to two frequencies, ω_+ and ω_-, which are near the average of the two uncoupled resonant frequencies, and whose difference will depend on the coupling. Our analysis will ignore the series resistances in the two circuits since we already know that a small amount of damping will not significantly affect the resonant frequencies.

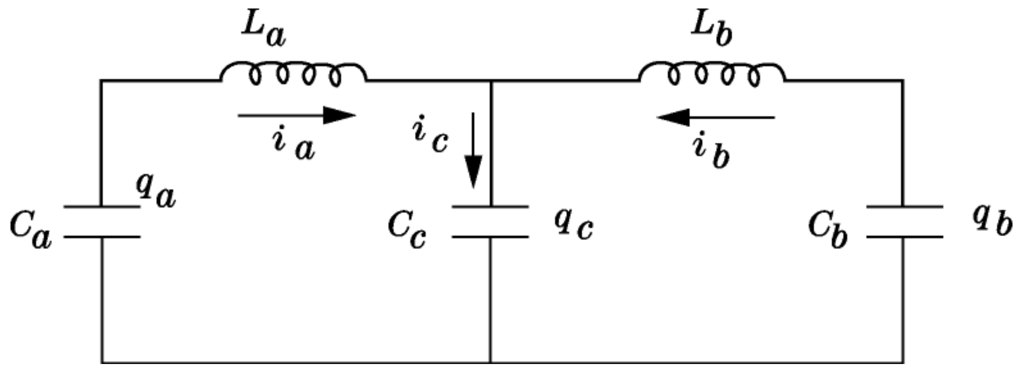

Fig 2: Schematic diagram of coupled RLC circuits.

From Fig. 2, we can write the equations for the currents in the two loops and the charges on the three capacitors as

$$\frac{q_a}{C_a} - L_a \frac{di_a}{d_t} - \frac{q_c}{C_c} = 0 \tag{1}$$

$$\frac{q_b}{C_b} - L_b \frac{di_b}{dt} - \frac{q_c}{C_c} = 0 \tag{2}$$

where

$$i_a = -\frac{dq_a}{dt} \tag{3}$$

$$i_b = -\frac{dq_b}{dt} \tag{4}$$

$$i_C = -\frac{dq_c}{dt} = i_a + i_b \tag{5}$$

The last equation implies that

$$q_c = -(q_a + q_b) \tag{6}$$

If we define two new frequencies ω_a and ω_b by

$$\omega_a^2 = \frac{1}{L_a}\left(\frac{1}{C_a}+\frac{1}{C_c}\right) \quad \omega_b^2 = \frac{1}{L_b}\left(\frac{1}{C_b}+\frac{1}{C_c}\right) \tag{7}$$

and two coupling constraints ω_{ac} and ω_{bc} by

$$\omega_{aC}^2 = \frac{1}{L_a C_c} \qquad \omega_{bC}^2 = \frac{1}{L_b C_c} \tag{8}$$

we can derive the following two linear second order differential equations

$$\ddot{q}_a + \omega_a^2 q_a + \omega_{aC}^2 q_b = 0 \tag{9}$$

$$\ddot{q}_b + \omega_b^2 q_a + \omega_{bC}^2 q_a = 0 \tag{10}$$

These are the homogeneous equations whose solutions will give us the natural frequencies of oscillation of the coupled systems. We find the solutions by assuming the form

$$q_a(t) = A_a e^{t\omega+t} + B_a e^{t\omega-t} \tag{11}$$

$$q_b(t) = A_b e^{t\omega+t} + B_b e^{t\omega-t} \tag{12}$$

Substituting and equating coefficients of $e^{i\omega} \pm^{t}$ gives four linear equations.

$$-\omega_+^2 A_a + \omega_a^2 A_a + \omega_{aC}^2 A_b = 0 \tag{13}$$

$$-\omega_-^2 B_a + \omega_a^2 B_a + \omega_{aC}^2 B_b = 0 \tag{14}$$

$$-\omega_+^2 A_b + \omega_b^2 A_b + \omega_{bC}^2 A_a = 0 \tag{15}$$

$$-\omega_-^2 B_b + \omega_b^2 B_b + \omega_{bC}^2 B_a = 0 \tag{16}$$

Solving, for example, the equations for A_a and A_b, we get

$$A_a = \frac{\omega_+^2 - \omega_b^2}{\omega_{bC}^2} A_b \qquad (17)$$

$$A_b = \frac{\omega_+^2 - \omega_a^2}{\omega_{aC}^2} A_a \qquad (18)$$

from which we eliminate the A's to obtain a quadratic in ω+2.

$$\left(\omega_+^2 - \omega_a^2\right)\left(\omega_+^2 - \omega_b^2\right) - \omega_{aC}^2 \omega_{bC}^2 = 0 \qquad (19)$$

The B equations lead to the same relation for ω-2, so the two solutions of Eq. 19 can be identified as ω+ and ω-. The angular frequencies are, therefore

$$\omega_\pm^2 = \frac{1}{2}\left(\omega_a^2 + \omega_b^2\right) \pm \frac{1}{2}\left[\left(\omega_a^2 - \omega_b^2\right)^2 + 4\omega_{aC}^2 \omega_{bC}^2\right]^{1/2} \qquad (20)$$

where ω+ has the plus sigh and the ω- sign. The general solutions for the change are

$$q_a = A_a e^{t\omega + t} + B_a e^{t\omega - t} \qquad (21)$$

$$q_b = A_a \frac{\omega_+^2 - \omega_a^2}{\omega_{aC}^2} e^{t\omega + t} + B_a \frac{\omega_-^2 - \omega_a^2}{\omega_{aC}^2} e^{t\omega - t} \qquad (22)$$

for arbitrary A_a and B_a. (The physical solutions are, of course, the real parts of Eq. 21 and 22.) The resultant motion can obviously be quite complicated.

As a brief digression, we note that the solution can be conceptually simplified by identifying the normal modes. By definition, the normal modes x_1 and x_2 are linear combinations of q_a and q_b which oscillate at ω_+ or ω -, rather than at both frequencies. Upon inspection of Eq. 21 and 22, one finds that the following work:

$$x_1 = q_a - \frac{\omega_{aC}^2}{\omega_-^2 - \omega_a^2} q_b = A_a \left[1 - \frac{\omega_+^2 - \omega_a^2}{\omega_-^2 - \omega_a^2}\right] e^{t\omega + t} \qquad (23)$$

$$x_2 = q_a - \frac{\omega_{aC}^2}{\omega_+^2 - \omega_a^2} q_b = B_a \left[1 - \frac{\omega_-^2 - \omega_a^2}{\omega_+^2 - \omega_a^2}\right] e^{t\omega - t} \qquad (24)$$

Notice that now the time independence is only $e^{i\omega}+^{t}$ for $^{x}1$ and $e^{i\omega}t$ for x_2, as required. Furthermore the amplitudes are given by either Aa or Ba, but not as a linear combination of the two. Thus the motion has been decomposed into two simple harmonic oscillations similar to the case of the pendulum discussed above. While this is a very pretty result, it is unfortunately not directly observable in our electrical circuit.

What we can measure are the voltages on the capacitors, which are proportional to q_a and q_b respectively. It is possible to get a better grasp of what those oscillations should look like by rewriting the solutions in terms of the average of the two normal mode frequencies

$$\overline{\varpi} = \tfrac{1}{2}(\omega_+ + \omega_-) \qquad (25)$$

and their difference

$$\omega_{ex} = \tfrac{1}{2}(\omega_+ - \omega_-) \qquad (26)$$

When these definitions are substituted into Eq. 21 and 22 we get

$$q_a = e^{t\varpi t}\left[A_a e^{t\omega_{ex}t} + B_a e^{-t\omega_{ex}t}\right] \qquad (27)$$

$$q_b = e^{t\varpi t}\left[A_a \frac{\omega_+^2 - \omega_a^2}{\omega_{aC}^2} e^{t\omega_{ex}t} + B_a \frac{\omega_-^2 - \omega_a^2}{\omega_{aC}^2} e^{t\omega_{ex}t}\right] \qquad (28)$$

Although these equations are still quite messy, we can qualitatively conclude that for large C_c values where $\varpi >> \omega_{ex}$, we have a rapid oscillation with a complicated but slower envelope. (In the real system there will be an additional exponential decrease, due to the inevitable resistive losses in the circuits.)

The equations become much simpler for the special case of identical oscillators. Then $L_a = L_b = L$ and $C_a = C_b = C$ so Eq. 20 reduces to

$$w_+^2 = \frac{1}{LC} + \frac{2}{LC_C} \qquad (29)$$

$$w_-^2 = \frac{1}{LC} \qquad (30)$$

where we also used definitions in Eq. 7 and 8. With these relations, the expression for q_b simplifies to

$$q_b = e^{t\varpi t}\left[A_a e^{t\omega_{ex}t} - B_a e^{-t\omega_{ex}t}\right] \qquad (31)$$

Further simplification occurs if we choose initial conditions such that $q_a(0)$ is non-zero, while $q_b(0)$ and all the currents are zero. (In the pendulum example, this is equivalent to pulling one mass aside and releasing it from rest.) Then $A_a = B_a = A$ and we obtain

$$q_a = Ae^{t\varpi t}\left[e^{t\omega_{ex}t} + e^{-t\omega_{ex}t}\right] \tag{32}$$

$$q_b = Ae^{t\varpi t}\left[e^{t\omega_{ex}t} - e^{-t\omega_{ex}t}\right] \tag{33}$$

Expanding the exponentials into sines and cosines and taking the real parts we finally arrive at

$$q_a = -2A\cos\varpi t\cos\omega_{ex}t \tag{34}$$

$$q_b = -2A\sin\varpi t\sin\omega_{ex}t \tag{35}$$

This shows quite clearly that q_a and q_b each consist of a high frequency oscillation at ϖ modulated by a lower frequency ω_{ex}. The modulation on q_a and q_b is ninety degrees out of phase, indicating that the total energy is flowing back and forth between the circuits. Comparing these results with what we found in the more general cases we can recognize the claimed similarities in behavior. You will also have a chance to see all of this experimentally. For later use we also derive an explicit expression for w_{ex} for identical oscillators in the limit ω_{ac}, $\omega_{bc} << \omega+, \omega-$:

$$\omega_{ex} = \frac{C}{2C_C}\sqrt{\frac{1}{LC}} = \frac{C}{2C_C}\varpi \tag{36}$$

We can calculate this from component values and compare with a measured. ω_{ex}.

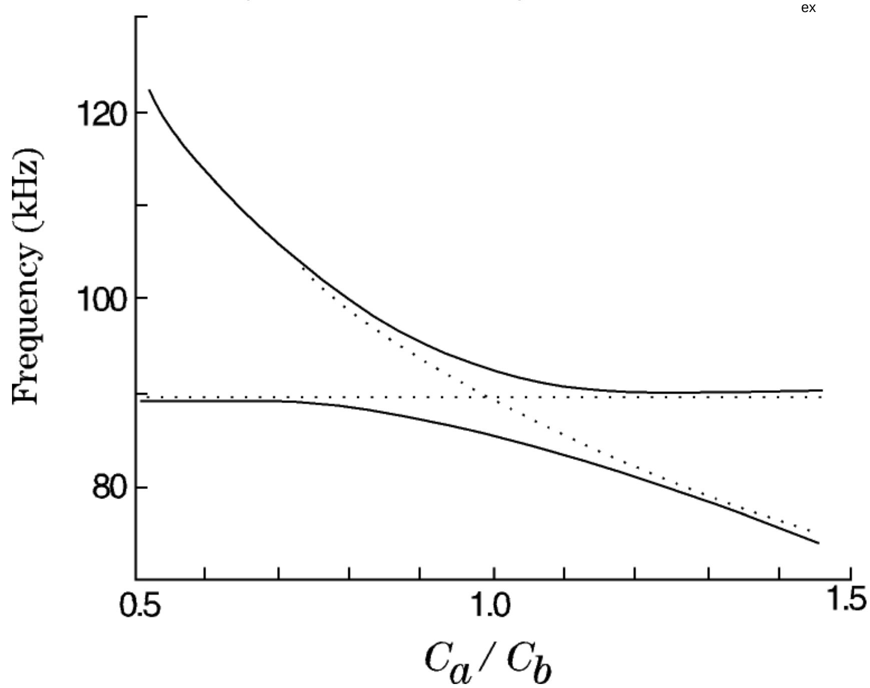

Fig. 3 Normal mode frequencies (solid lines) as a function of capacitance ratio. The dotted lines are ω_a and ω_b in the absence of coupling.

In principle we could find the normal mode frequencies for arbitrary component values using the free oscillations described by Eq. 27 and 28, but in practice they are very hard to determine from the complicated waveform. Instead, we can drive one oscillator with a sine wave and measure the response of the coupled system. We expect, from previous experiments, that the response of the circuit will be maximum at the natural frequencies, so this gives us a way to measure $\omega+$ and angular frequency $\omega-$ directly. The resulting data should look something like Fig. 3, which is a plot of Eq. 20 for experimentally reasonable parameters. Note that as the circuits are tuned toward equal resonant frequency the normal mode frequencies come together but are never equal.

Experimental Procedure:

A. Energy Modulation

Our first exercise will be to pulse the circuits and watch the energy slosh back and forth between the two capacitors. The method is similar to what you used in studying transients in a single RLC circuit.

Wire up the circuit shown in Fig. 4. The variable capacitor has a range of 250 - 650 pF so it can vary the oscillation frequency for the first RLC circuit in a range centered on that of the second. The 3.3 ohm resistor is used to reduce the impedance of the function generator. Otherwise the 50 ohm internal impedance would introduce too much damping in our first oscillator. Channel 1 measures the charge on C_a (approximately) and Channel 2 measures the charge on C_b. Begin with C_c at an intermediate value, say 0.01 µF.

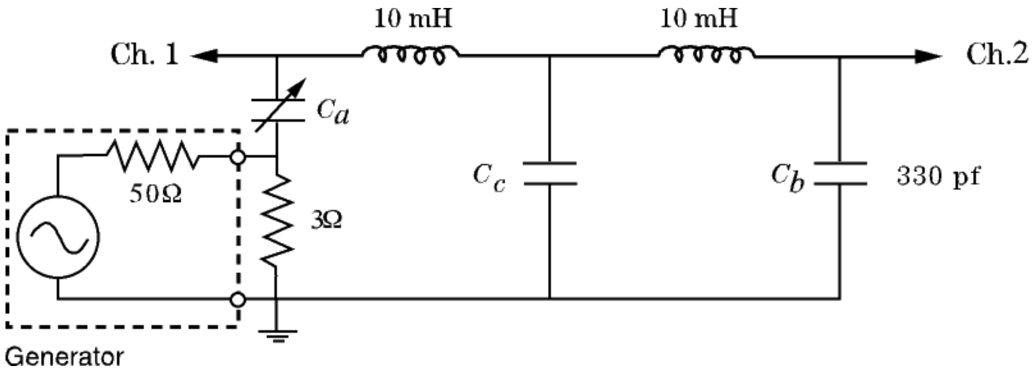

Fig. 4 Experimental arrangement for studying coupled RLC circuits. To minimize interference, connect the metal case of the circuit board to the scope ground.

Set the Agilent 33120A function generator to produce a maximum-amplitude square wave at 200-300 Hz. Adjust the time base and vertical sensitivity of the scope so that you can see the transients that occur each time the square wave reverses polarity. You will probably find it convenient to connect the Agilent 33120A's "TRIG" output as an external trigger for the scope.

Vary the tunable capacitor and observe the qualitative behavior of the signals. Find the capacitor setting that gives the slowest beat frequency on C_b. By minimizing ω_{ex} in this way we achieve the condition $C_a = C_b$ assumed in our previous analysis. Using this setting, sketch the two waveforms and identify the relevant features on your sketch. You should be able to recognize the decay of the signal due to the finite resistance in the circuit and the expected $\cos \omega_{ex}t$ and $\sin \omega_{ex}t$ dependencies of the q_a and q_b envelopes. Be sure to get a good estimate of the modulation frequency ω_{ex} for comparison with later measurements. Don't forget that the minima in the modulation envelope occur at intervals of $2\pi / 2\omega_{ex}$, not $2\pi / \omega_{ex}$.

After doing the detailed sketches for C_c = 0.01 µF, describe the changes when you use a 0.0047 µF and a 0.047 µF coupling capacitor. As before, adjust the variable capacitor setting to obtain the slowest beat.

B. Normal frequencies

In this part we will use the function generator to excite the first oscillator with a sinusoidal wave and measure the response of the second oscillator. The scope is used to measure the voltage and hence infer the charge on capacitors C_a and C_b.

Reset the function generator to produce sine waves at a frequency near the expected resonance frequency. Install the 0.01 µF coupling capacitor and set the variable capacitor to a convenient dial setting. (Unfortunately the capacitance and dial setting are not linearly related.) If you now slowly vary the function generator frequency through a range around the expected resonance you will notice two relative maxima in the voltage across C_b. These maxima occur at frequencies corresponding to *angular frequency*+ and angular frequency -.

You can easily visualize what is happening by using the function generator to automatically sweep the driving frequency sweep rate. Set at 5-10 Hz . If the driving generator is set near the resonance frequency you should see two peaks on the display. Adjust the amplitude and offset of the sweep generator and the scope scales to get a nice display.

Using the swept-frequency display, describe what happens as you adjust the variable capacitor. Are your observations consistent with Fig. 3? What changes when you use the 0.0047 µF and 0.047 µF capacitors instead of the 0.01 µF?

More quantitative results can be obtained by turning off the sweep function and measuring ω_+ and ω_- for the complete range of the variable capacitor. Graph the two frequencies versus dial setting on one plot for the 0.01 µF coupling capacitor. Compare the minimum splitting you find between ω_+ and ω_- with the values you found from the beat envelope in part A and from the component values using Eq. 36.

Report:
The report will consist of the various sketches and observations, the graph of normal mode frequencies versus variable capacitor dial setting, and the comparison of the values of ω_{ex} from Parts A and B.